PhD defence
Higher commutativity in algebra and algebraic topology

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Introduction to higher structures

What is algebraic topology?

- Originates in the work of Poincaré.

Figure: Henri Poincaré (1854-1912)

- The aim is to understand the **shape** and **form** of topological spaces using algebraic invariants with the goal of distinguishing them up to **homeomorphism** or **homotopy equivalence**.
- The first algebraic invariant is number of holes (**homotopy** or **(co)homology groups**), but this both a) too difficult and b) insufficient. We need more structure.

What is higher commutativity?

1. The integers are equipped with a commutative multiplication $2 \times 3 = 3 \times 2$.
2. Similarly spaces can also be equipped with various (co)multiplications. For example, one always has the diagonal map.

$$X \to X \times X$$

$$x \mapsto (x, x)$$

Based loop spaces $\mathrm{Map}_* \left(S^1, X\right) = \Omega\left(X\right)$ are also be equipped with **loop concatenation**:

$$\Omega\left(X\right) \times \Omega\left(X\right) \to \Omega\left(X\right)$$

This is **homotopy associative**, ie. $\pi_1(X)$ is a group. If you take $\Omega^2\left(X\right)$ it becomes **homotopy commutative** - ie. $\pi_i(X)$ is a commutative group for $i > 0$.

Algebras

Definition

A **dg-algebra** is a chain complex A equipped with a binary associative multiplication $- \cup - : A^p \otimes A^q \to A^{p+q}$ and d is a derivation wrt. $\cup$

$$d(x \cup y) = d(x) \cup y + (-1)^{|x|} x \cup d(y)$$

Example: if you have a smooth manifold M, the de Rham forms $(\Omega^\bullet(M, \mathbb{R}), \wedge)$ form a **commutative dg-algebra**.

Weak equivalence of algebras

Definition

Two dg-algebras A, B are **weakly (homotopy) equivalent** if they can be linked via a zig-zag of algebras where all the maps are cohomology equivalences.

$$A \xrightarrow{\sim} X \xleftarrow{\sim} \dots \xrightarrow{\sim} Y \xleftarrow{\sim} B$$

Example: if you have a smooth manifold M, the de Rham forms $(\Omega^\bullet(M, \mathbb{R}), \wedge)$ are weakly equivalent to $(C^*(X, \mathbb{R}), \cup)$. This is one of the two central ideas of **Sullivan's approach to rational homotopy theory.** This does not hold when the coefficient ring is not a field of characteristic 0.

Operads

Definition

An **operad** $\mathcal{P}$ in a monoidal category C is a collection of objects $\mathcal{P}(n) \in C$. Each object $\mathcal{P}(n)$ is equipped with an action of the symmetric group $\mathbb{S}_n$ and there is a composition law

$$\mathcal{P}(n) \circ_i \mathcal{P}(m) \to \mathcal{P}(n+m-1)$$

Example

The **endomorphism operad**

$$\mathrm{End}(X)(n) = \mathrm{Map}(X^{\times n}, X)$$

The composition law is given by

$$\mathrm{End}(X)(n) \circ_i \mathrm{End}(X)(m) \to \mathrm{End}(X)(n+m-1)$$

$$(f, g) \mapsto (\mathrm{id} \times \mathrm{id} \times \cdots \times g \times \mathrm{id} \times \cdots \times \mathrm{id}) \circ f$$

Algebras over operads

Definition

An **algebra over an operad** $\mathcal{P}$ is an object $X \in C$ and a map of operads

$$\mathcal{P} \to \mathrm{End}(X)$$

Examples

- There an operad for associative algebras $\mathrm{Ass}(n) = R[\mathbb{S}_n]$.
- There an operad for commutative algebras $\mathrm{Com}(n) = R$.
- There is an infinite family of operads, each equipped with a free action of the symmetric group interpolating between the two

$$\mathrm{Ass} \xleftarrow{\sim} E_1 \subset E_2 \subset \cdots \subset E_\infty \xrightarrow{\sim} \mathrm{Com}$$

Mandell's theorem

The singular cochain complex $C^\bullet(X, R)$ can, via explicit formulae given by Berger and Fresse, be equipped with the structure of an E_∞-algebra.

Theorem (Mandell, 2003)

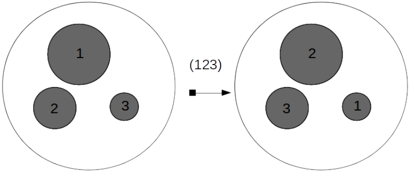
Two finite type, nilpotent spaces X and Y are weakly equivalent and only if their E_∞-algebras of singular cochains with integral coefficients are quasi-isomorphic as E_∞-algebras.

Recognition and corecognition

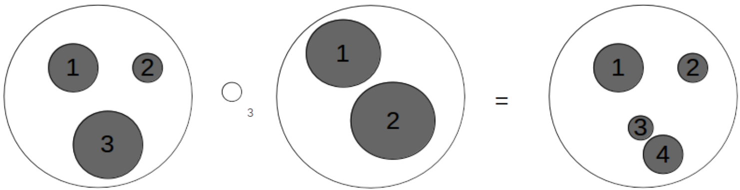
What does an E_n operad look like?

Arity k-component of the little n-disc operad $\mathbb{D}_n$

- Start with the standard n-disc.
- Consider the space of all pairwise disjoint embedding of k smaller n-discs into it
- These discs are labelled $\{1, \ldots, k\}$
- Symmetric group acts by permuting the labels.



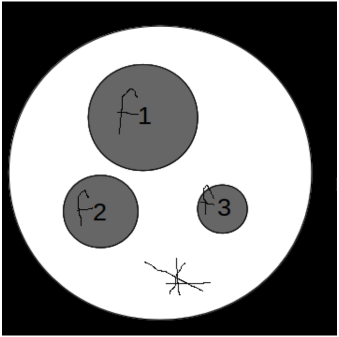
The little n-disc operad: operadic composition



Action on loop spaces

- An n-fold loop space is a space of the form $\mathrm{Map}_*(S^n, X)$.
- You have an action

$$\mathcal{D}(n) \times \mathrm{Map}_*(S^n, X)^{\times n} \to \mathrm{Map}_*(S^n, X)^{\times n}.$$



- This generalises loop concatenation.

May's Recognition Principle

Theorem (May [2], 1972)

Every n-fold loop space is a $\mathcal{D}_n$-algebra, and if a pointed grouplike space is a $\mathcal{D}_n$-algebra then it has the weak homotopy type of an loop space.

- Opened the door to the computation of $H_*(\Omega^n X)$
- Significant to the development of **stable homotopy theory**.

The dual principle

- The **smash product** of two pointed spaces $X \wedge Y$ is

$$(X \times Y)/(* \times Y \cup X \times *)$$

- An n-**fold reduced suspension** $\Sigma^n X = S^n \wedge X$.

Theorem (FC, Moreno-Fernández, Wierstra)

Every n-fold reduced suspension is a $\mathcal{D}_n$-coalgebra, and if a pointed space is a $\mathcal{D}_n$-coalgebra then it is homotopy equivalent to an n-fold reduced suspension.

- This is the Eckmann-Hilton dual to May's theorem.
- The key step in the proof is to precisely describe the comonad associated to an operad in pointed topological spaces.
- There is a corecognition principle already for coalgebras over the comonad $\Sigma^n \Omega^n$. These are all suspensions on the nose.

Coalgebras over an operad

Example

The **coendomorphism operad**

$$\mathrm{CoEnd}(X)(n) = \mathrm{Map}(X, X^{\vee n})$$

The composition law is given by

$$\mathrm{CoEnd}(X)(k) \times \mathrm{CoEnd}(X)(n_1) \times \cdots \times \mathrm{CoEnd}(X)(n_k) \xrightarrow{\circ} \mathrm{CoEnd}(X)(n_1 + \cdots n_k)$$

$$(f; f_1, \ldots f_k) \mapsto f \circ (f_1 \vee f_2 \vee \cdots \vee f_k)$$

Definition

An **coalgebra over an operad** $\mathcal{P}$ is an object $X \in C$ and a map of operads

$$\mathcal{P} \to \mathrm{CoEnd}(X)$$

Coalgebras in pointed spaces

Example

The pinch map equips the n-sphere S^n with the structure of a coalgebra over the little n-discs operad. More generally n-fold suspensions $\Sigma^n X = S^n \wedge X$ are too.

- The category of $\mathcal{P}$-coalgebras in spaces turns out to be the **co-Eilenberg-Moore category** of a certain comonad $C_{\mathcal{P}}$.
- This comonad is much smaller than you might expect.

Example (Failure of Eckmann-Hilton duality)

An explicit description of this comonad shows that there are no non-trivial strictly commutative or strictly coassociative algebras in spaces. So equivalent operads **do not** give rise to equivalent categories of coalgebras.

Higher cohomology operations

Barebones Massey product formalism

Given an algebraic model category M where all objects are fibrant, one constructs the **matric Massey products** for an object $x \in M$ as follows.

- One takes a functorial cofibrant replacement $m(x)$ for x.
- Suppose $m(x)$ admits a functorial filtration (generally by some notion of weight).
- Suppose that that E_1-page of the associated spectral sequence depends only on $H(x)$
- Then there are chain-level descriptions of the higher differentials, via the staircase lemma, that are homotopy invariant.

Massey triple products

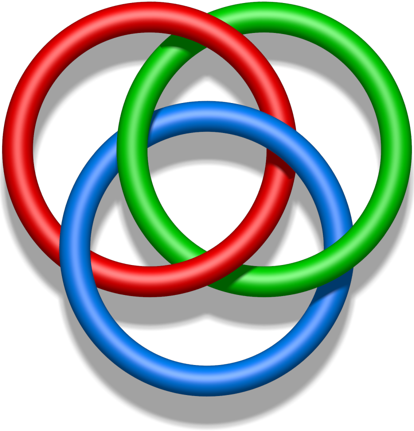
Definition

Let A be a dg-algebra. Let $a, b, c \in H^\bullet(A)$ by such that $ab = 0$ and $bc = 0$. Let x, y, z be cocycles representing a, b, c and suppose $du = xy$ and $dv = yz$. Then $uz - xv$ is a cocycle and represents a well-defined class of

$$\frac{H^{|a|+|b|+|c|-1}(A)}{aH^{|b|+|c|-1}(A) + H^{|a|+|b|-1}(A)c}$$

Muro recently generalised Massey triple products to $\mathcal{P}$-algebras over a quadratic operad [3].

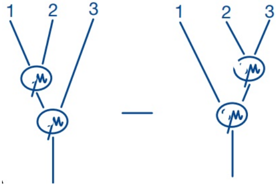
The geometric picture



Source: Jim.belk; Wikipedia

Operads and trees

- The associative operad is generated by a single arity two operation $\mu = - \cdot - \in \mathcal{P}(2)$
- The free operad $\mathcal{F}(R)$ is made up of sums of trees.
- To get the associative operad we quotient $\mathcal{F}(\mu)$ by an operadic ideal generated by the following element.



- The associative operad is $\mathcal{F}(R)/(E)$, it is *quadratic.*

Koszul duality

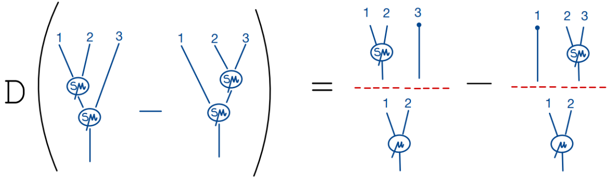
- For *quadratic operads*, one has a Koszul dual cooperad $\mathcal{P}^{\text{i}}$

$$(\mathcal{F}(R)/(E))^{\text{i}} = \mathcal{C}(sR, s^2E) \hookrightarrow \mathcal{F}^c(sR)$$

- This also admits a description in terms of trees.
- In nice situations, when $\mathcal{P}$ is **Koszul**, one has that $B\mathcal{P}^{\text{i}} \xrightarrow{\sim} \mathcal{P}$ is a minimal model.
- This relationship, **Koszul duality**, is both reciprocal and ubiquitous in nature. Ass $\sim$ Ass, Pois $\sim$ Pois, Lie $\sim$ Com, Leibniz $\sim$ Zinbiel. There are examples of non-Koszul operads like PreLie $\sim$ Perm.

Generalising Massey products

- For Koszul $\mathcal{P}$, the primitive $\mathcal{P}$-Massey products correspond precisely to co-operations, represented as trees, in the Koszul dual cooperad $\mathcal{P}^{\text{i}}$. The order of the operation corresponds to the weight of the tree.
- You have an inductive map on the weight of the tree given by pruning all the branches at the root of trees.



- The weight zero operation correspond to the initial inputs.

Theorem (FC-Moreno-Fernandez, 2023)

Weakly equivalent $\mathcal{P}$-algebras have the same Massey products.

Examples

Specializing to various cases, we recover:

- Weight 1 trees: regular operations on the $\mathcal{P}$-algebra.
- Associative operad: classical Massey products
- Lie operad: The Lie-Massey brackets of Retah and Alladay
- Weight 2 trees: Muro's generalisations of Massey triple products.
- Dual numbers operad $\mathcal{D}$: Algebras over $\mathcal{D}$ are bicomplexes. The Massey products are precisely the differentials in the associated spectral sequence.
- Poisson operad: Messy formulae.

Other properties of Massey products

- Given a morphism of operads $f : \mathcal{P} \to \mathcal{Q}$, one has induced functors on the Eilenberg-Moore categories.

$$f_! : \mathcal{P} - \mathrm{Alg} \leftrightarrows \mathcal{Q} - \mathrm{Alg} : f^*.$$

  With some technical assumptions, Massey products can be pushed and pulled between these categories via these functors.
- Given an $\mathcal{P}$-algebra A and a choice of homotopy retract onto its homology, by the **homotopy transfer theorem** there is a quasi-isomorphic $\mathcal{P}_\infty$-structure on $H(A)$.
  1. For any $\mathcal{P}$-Massey product in $x \in < x_1, \cdots x_n >$, one can always find a P_∞-structure on H and higher multiplication m satisfying $m(x_1, \cdots, x_n) = x$.
  2. But for a random P_∞-structure on H , the higher multiplication $m(x_1, \cdots, x_n)$ will not generally be a Massey product - the lower multiplications must be trivial in a very specific way.

Cotriple products

Question

What are the $\mathcal{P}$-Massey products over $\mathbb{F}_p$?

- The $\mathcal{P}$-Massey products still work.
- Over $\mathbb{F}_p$, the bar-cobar resolution no longer completely works.
- So one uses the cotriple cofibrant replacement and filter using the skeletal filtration.
- We call the resulting operations **cotriple products.**
- For the commutative operad, the secondary cotriple operations turn out to be easy to calculate.

Applications: Producing counterexamples

Cotriple products can be used to produce examples of:

- Commutative algebras A, B over $\mathbb{Z}$ without torsion in their cohomology such that $A \otimes \mathbb{Q}$ and $B \otimes \mathbb{Q}$ are weakly equivalent, but $A \otimes \mathbb{F}_p$ and $B \otimes \mathbb{F}_p$ are not.
- Commutative algebras which have a divided power structure on cohomology but which are not weakly equivalent to a divided power algebra.
- Commutative algebras A, B over $\mathbb{F}_p$, which are weakly equivalent as associative algebras but not commutative algebras. This answers a question raised in a recent paper[1].
- Commutative algebras A, B over $\mathbb{F}_p$ that are weakly equivalent as E_∞-algebras but not commutative algebras.

[1] R. Campos et al. *Lie, associative and commutative quasi-isomorphism*. To appear in Acta Mathematica. arXiv: 1904.03585 [math.RA].

Rectification

Question

When is an E_∞-algebra is weakly equivalent to a commutative algebra?

There are obstructions: A subset of the cotriple operations, called **higher Steenrod operations**. These include all of the Steenrod operations except $\mathrm{Sq}^n(x)$ when $|x| = n$ + higher obstructions.

Theorem (F.C.)

An E_∞-algebra is rectifiable if and only if its higher Steenrod operations vanish coherently.

Other work

p-adic de Rham forms

- In 1972, Sullivan [5] defined the **the algebra of piecewise linear differential forms**: essentially a generalization of the de Rham forms functor to arbitrary simplicial sets. This is a strictly commutative algebra.
- The limitation of this approach is that it only works in zero characteristic.
- Using divided power algebras, one can construct a similar functor $\Omega^*(X, \widehat{\mathbb{Z}_p})$ that approximates *some* of the information about the homotopy type of E_∞-algebra $C^*(X, \widehat{\mathbb{Z}_p})$.
- The information in question is all of the cohomology, most of the Massey products and coherence data.
- The p-adic de Rham forms are weakly equivalent to $\eta\left(C^*\left(X, \widehat{\mathbb{Z}_p}\right)\right)$ where η is a *décalage* functor occurring in crystalline cohomology.

Higher Hochschild-Kostant-Rosenberg Theorem

Theorem (Hochschild-Kostant-Rosenberg Theorem)

Let $\mathbb{k}$ be a field of characteristic 0 and let A be a commutative $\mathbb{k}$-algebra which is essentially of finite type and smooth over $\mathbb{k}$. Then there is an isomorphism of graded $\mathbb{k}$-algebras

$$\Phi : HH_*(A, A) \xrightarrow{\sim} \Omega^*(A, \mathbb{k})$$

between the Hochschild homology and the module of Kähler differentials.

The higher Hochschild-Kostant-Rosenberg theorem

The Hochschild chain complex $C_*(A, A)$ is intuitively 'circle'-shaped. Pirashvili [4] has generalised this to more general complex $A \boxtimes X$ for any simplicial set X.

Theorem

Let X be a formal simplicial set of finite type in each degree. Let A be a CDGA. Suppose that $(\mathrm{Sym}(V), d)$ is a cofibrant, quasi-free resolution of A. Then there is a natural equivalence of chain complexes

$$A \boxtimes X \xrightarrow{\sim} \mathrm{Sym}(V \otimes H_*(X), d_X)$$

This equivalence is functorial with respect to formal maps.

- When $X = \Sigma^n X$, one can explicitly construct a homotopy Pois_n-structure on the left hand side. This is equivalent to the Deligne conjecture by abstract nonsense.

Horizons

Further questions

1. Find a precise statement for Eckmann-Hilton duality over $\mathbb{Z}$ akin to Koszul duality in characteristic 0.
2. Are divided power algebras A and B quasi-isomorphic as divided power algebras if and only if they are quasi-isomorphic as associative algebras?
3. Study Massey products in other situations:
   - Relate them directly to the more general phenomenon of (non-operadic) Koszul duality
   - Use them to study algebras over modular operads or even modular operads or graph complexes themselves, where one would need to generalise from rooted trees to more general graphs.

References I

[1] R. Campos et al. *Lie, associative and commutative quasi-isomorphism*. To appear in Acta Mathematica. arXiv: 1904.03585 [math.RA].

[2] J. P. May. *The geometry of iterated loop spaces*. Lectures Notes in Mathematics, Vol. 271. Springer-Verlag, Berlin-New York, 1972, pp. viii+175.

[3] F. Muro. "Massey products for algebras over operads". In: (). To appear in *Communications in Algebra*, 2023.

[4] T. Pirashvili. "Hodge decomposition for higher order Hochschild homology". In: *Ann. Sci. École Norm. Sup. (4)* 33.2 (2000), pp. 151–179. ISSN: 0012-9593. DOI: 10.1016/S0012-9593(00)00107-5. URL: https://doi.org/10.1016/S0012-9593(00)00107-5.

References II

[5] D. Sullivan. "Infinitesimal computations in topology". In: *Inst. Hautes Études Sci. Publ. Math.* 47 (1977), 269–331 (1978). ISSN: 0073-8301.